Calculation of the Effective Coefficients

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1 Introduction

Let us consider the operator

$$\mathcal{L} \equiv -\nabla \cdot A(y)\nabla, \tag{1}$$

where $A(y) = [a_{ij}(y)]_{1 \le i,j \le N}$ is symmetric and satisfies $\alpha|\xi|^2 \le \xi_i a_{ij} \xi_j \le \beta|\xi|^2$ for all $\xi \in \mathbb{R}^N$ with $0 < \alpha < \beta$. Moreover, $a_{ij}(y)$ are Y-periodic positive functions, where $Y = (-1/2, 1/2)^d$ and $a_{ij}(y) \in L^\infty(Y)$. In other words, $a_{ij}(y)$ are $1-$periodic functions. Let us assume that the reference cell Y consists of two isotropic materials. One is a matrix part Y_M and the other is a fiber part Y_F such that $Y = Y_M \cup Y_F$ and $\partial Y_M \cap \partial Y_F = \Gamma$ (See the Figure 2.1). Then the coefficient matrix $[a_{ij}(x)]$ satisfies the followings:

$$[a_{ij}(y)] = \begin{cases} a^M I, & y \in Y_M \\ a^F I, & y \in Y_F, \end{cases} \tag{2}$$

where I is an identity matrix and a^F, a^M are positive constants. For each ϵ, we define the operator $\mathcal{L}^\epsilon$ by

$$\mathcal{L}^\epsilon \equiv -\nabla \cdot A^\epsilon(x)\nabla \text{ with } A^\epsilon(x) = A(\frac{x}{\epsilon}). \tag{3}$$

Let us note that $a_{ij}(x/\epsilon)$ are ϵ-periodic functions. Now we can consider the model problem associated with $\mathcal{L}^\epsilon$

$$\mathcal{L}^\epsilon u^\epsilon = f \quad \text{in} \quad \Omega, \qquad u_\epsilon = 0 \quad \text{on} \quad \partial\Omega, \tag{4}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^d$ and $f \in L^2(\Omega)$.

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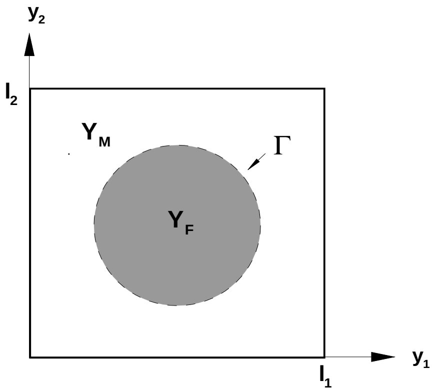


Fig. 1. The Reference cell Y in $\mathbb{R}^2$.

Let u^* be the solution of the homogenized problem of (4):

$$\mathcal{L}^* u^* \equiv -\nabla \cdot (A^*(x)\nabla u) = f \quad \text{in } \Omega, \quad u^* = 0 \quad \text{on } \partial\Omega, \tag{5}$$

where the effective coefficient matrix $A^*(x) = [a^*_{ij}]$ is defined by

$$a^*_{ij} = \frac{1}{|Y|} \int_Y \left(a_{ik} \frac{\partial}{\partial y_k} \chi_j + a_{ij} \right) dy, \tag{6}$$

where χ_p is the $Y-$periodic solution of cell problem

$$-\mathcal{L}\chi_p = \frac{\partial}{\partial y_i} a_{ij}(y) \quad \text{in Y and} \quad \langle \chi_p \rangle \equiv \frac{1}{|Y|} \int_Y \chi_p(y) dy = 0. \tag{7}$$

Moreover, χ_j satisfies the continuous interface conditions along the Γ:

$$\chi_p(r-0,\theta) = \chi_p(r+0,\theta), \tag{8}$$

$$(a^F_{ij}\nu_j \frac{\partial \chi_p}{\partial y_i})(r-0,\theta) = (a^M_{ij}\nu_j \frac{\partial \chi_p}{\partial y_i})(r+0,\theta), \quad (r,\theta) \in \Gamma, \tag{9}$$

where ν_j is the j-th component of the normal vector ν on Γ. Let us define the space of periodic functions $W^1_{per}(Y)$ by

$$W^1_{per}(Y) = \{ v \in H^1_{per} | \langle v \rangle = 0 \}, \tag{10}$$

where $H^1_{per}(Y)$ is defined as the closure of $C^\infty_{per}(Y)$ for H^1-norm and $C^\infty_{per}(Y)$ is the subset of $C^\infty(\mathbb{R})$ of Y-periodic functions. Then we have the variational form of (7): Find $\chi_p \in W^1_{per}(Y)$ such that

$$\int_Y (\nabla(\chi_p + y_p))^T [a_{ij}(y)] \nabla v = 0, \quad \forall v \in W^1_{per}(Y). \tag{11}$$

Since $a_{pq}(y) = a_{pj}(y)\delta_{qj}$ and $\delta_{qj} = \dfrac{\partial y_q}{\partial y_j}$, where δ_{qj} is the Kronecker delta, the effective coefficients a^*_{pq} can be rewritten by

$$a^*_{pq} = \frac{1}{|Y|} \int_Y a_{ij}(y) \frac{\partial(\chi_q + y_q)}{\partial y_j} \frac{\partial y_p}{\partial y_i}. \tag{12}$$

By substituting a test function v in (11) by χ_p and by adding this to (12), we have

$$a^*_{pq} = \frac{1}{|Y|} \int_Y (\nabla(\chi_p + y_p)^T [a_{ij}(y)] \nabla(\chi_q + y_q). \tag{13}$$

Lemma 1 *Let a^*_{ij} and b^*_{ij} be the effective coefficients corresponding to the coefficients a_{ij} and $b_{ij}(y)$, respectively, where $b_{ij} = ka_{ij}(y)$ and k is constant. Then*

$$b^*_{ij} = ka^*_{ij}.$$

It is clear from the uniqueness of the solution of (7) and (13). Applying lemma 1, we can normalize the coefficient matrix in Y_M. Throughout the paper, the normalized coefficient in Y_M is considered. Let $[\kappa_{ij}(y)]$ be the material property matrix such that

$$\kappa_{ij}(y) = \begin{cases} \delta_{ij}, & y \in Y_M, \\ \kappa\delta_{ij}, & y \in Y_F, \end{cases} \tag{14}$$

where κ is constant. Let χ^κ_p be the solution of the cell problem (7) corresponding to $[\kappa_{ij}]$ and denote M^κ_p by $M^\kappa_p = \chi^\kappa_p + y_p$. Then, from (13) with the continuous interface condition, the effective coefficient κ^*_{pq} can be written by

$$\kappa^*_{pq} = \mathcal{B}_{Y_M}(M^\kappa_p, M^\kappa_q) + \kappa \mathcal{B}_{Y_F}(M^\kappa_p, M^\kappa_q), \tag{15}$$

where the bilinear form $\mathcal{B}_Y(u, v)$ is defined by

$$\mathcal{B}_Y(u, v) = \int_Y (\nabla u)^T \nabla v.$$

1.1 Basic properties for Effective coefficients

Here, in this section, we present the basic properties of the effective coefficients. More precisely, we show that the quantity of the effective coefficient depends on the material property on Y_F. In other words, the effective coefficient with bigger material property in Y_F is also bigger than the one with smaller material property in Y_M. For the most fiber reinforced composite materials, the fiber inclusion is symmetric with respect to a hyperplane. We assume that the reference cell $Y = (-1/2, 1/2)^d$ is symmetric with respect to the axis

$x = 0$ and $y = 0$. In a homogenization theory, it is well known that the effective coefficients κ^*_{ij} in the symmetric reference cell are constants. That is, $\kappa^*_{ij} = \kappa^* I$, where κ^* is a constant and I is the identity matrix.

For the simplicity purpose, M^κ_p is denoted by M^κ. Then, $M^\kappa - M^\eta = \chi^\kappa - \chi^\eta$. This implies that $M^\kappa - M^\eta \in W^1_{per}(Y)$. By substituting a test function v in (11) by $M^\kappa - M^\eta$ and using (15), we have the alternative definition for the effective coefficient as follows;

$$\kappa^* = \mathcal{B}_{Y_M}(M^\kappa, M^\eta) + \kappa \mathcal{B}_{Y_F}(M^\kappa, M^\eta). \tag{16}$$

Lemma 2 *For any positive constants κ, η, we have the followings*

$$\mathcal{B}_{Y_M}(M^\kappa, M^\eta) > 0, \quad \mathcal{B}_{Y_F}(M^\kappa, M^\eta) > 0.$$

Without loss of generosity, we can assume that $\kappa \neq \eta$. Since $M^\kappa - M^\eta \in W^1_{per}(Y)$, from (11), we have

$$\mathcal{B}_{Y_M}(M^\gamma, M^\kappa - M^\eta) + \gamma \mathcal{B}_{Y_F}(M^\gamma, M^\kappa - M^\eta) = 0, \quad \gamma = \kappa, \eta. \tag{17}$$

Subtracting equations for each $\gamma = \kappa, \eta$ yields

$$\mathcal{B}_{Y_M}(M^\kappa - M^\eta, M^\kappa - M^\eta) + \kappa \mathcal{B}_{Y_F}(M^\kappa, M^\kappa - M^\eta) - \eta \mathcal{B}_{Y_F}(M^\eta, M^\kappa - M^\eta) = 0. \tag{18}$$

Since the first term in the left hand side is positive, we have

$$\kappa \mathcal{B}_{Y_F}(M^\kappa, M^\kappa - M^\eta) - \eta \mathcal{B}_{Y_F}(M^\eta, M^\kappa - M^\eta) < 0,$$

which implies,

$$\kappa \mathcal{B}_{Y_F}(M^\kappa, M^\kappa) + \eta \mathcal{B}_{Y_F}(M^\eta, M^\eta) < (\kappa + \eta) \mathcal{B}_{Y_F}(M^\kappa, M^\eta). \tag{19}$$

This completes the proof of the second inequality from the fact that the left hand side is positive. In the similar manner, once can show the first inequality.

Now we are investigating the relation between the effective coefficients κ^* and η^* when $\kappa > \eta$. For $\gamma = \kappa, \eta$, (17) implies

$$\begin{aligned} \kappa^* &= \mathcal{B}_{Y_M}(M^\kappa, M^\eta) + \kappa \mathcal{B}_{Y_F}(M^\kappa, M^\eta). \\ \eta^* &= \mathcal{B}_{Y_M}(M^\eta, M^\kappa) + \eta \mathcal{B}_{Y_F}(M^\eta, M^\kappa). \end{aligned}$$

Subtracting above two equations yields

$$\kappa^* - \eta^* = (\kappa - \eta) \mathcal{B}_{Y_F}(M^\kappa, M^\eta). \tag{20}$$

Then from Lemma 1, (16) and (20) we have the following lemma.

Lemma 3 *If $\kappa > \eta$, then $\kappa^* > \eta^* > 0$.*

Remark 4 *In general, $\mathcal{B}_{Y_M}(M_p^\kappa, M_q^\eta) \not> 0$, $\quad \mathcal{B}_{Y_F}(M_p^\kappa, M_q^\eta) \not> 0$ because the first term of the left hand side in* (18) *is not positive. However, Lemma* 3 *is also true for the diagonal entries of the effective coefficients. It will be shown easily. In other words, $\kappa^*_{pp} > \eta^*_{pp}$ if $\kappa > \eta$.*

2 Effective coefficients for the summation of two material properties

Let a^*, b^* be the effective coefficients corresponding to the coefficient matrix

$$[\kappa^*(y)] = \begin{cases} I & \text{if } y \in Y_M, \\ \kappa I & \text{if } y \in Y_F, \end{cases} \tag{21}$$

where $\kappa = a, b$. In order to obtain the effective coefficient $c^* = (a+b)^*$ corresponding to the coefficient matrix, $[(a_{ij} + b_{ij})(y)]$, the cell problem must be solved and auxiliary calculation is required. In this section, we will show that c^* can be expressed in terms of the effective coefficients a^*, b^* and constants a, b.

Then, from (17), we have the following representations of effective coefficients a^*, b^*, and $c^* = (a+b)^*$ for $\eta = a, b$ and $c = a + b$:

$$a^* = \mathcal{B}_{Y_M}(M^a, M^\eta) + a\mathcal{B}_{Y_F}(M^a, M^\eta), \tag{22}$$
$$b^* = \mathcal{B}_{Y_M}(M^a, M^\eta) + b\mathcal{B}_{Y_F}(M^a, M^\eta), \tag{23}$$
$$c^* = 2\mathcal{B}_{Y_M}(M^c, M^\eta) + c\mathcal{B}_{Y_F}(M^c, M^\eta)(\text{non-normalized form}). \tag{24}$$

In the following lemma, the effective coefficient c^* can be expressed in the several forms in terms of a^*, b^*, a, and b.

Lemma 5

$$\begin{aligned} (i) \quad & c^* = a^* + b^* + \frac{b-a}{2}\mathcal{B}_{Y_F}(M^c, M^a - M^b), \\ (ii) \quad & c^* = 2a^* + (b-a)\mathcal{B}_{Y_F}(M^c, M^a), \\ (iii) \quad & c^* = 2b^* + (a-b)\mathcal{B}_{Y_F}(M^c, M^b). \end{aligned}$$

In particular, if $a = b$, then $(a+b)^* = a^* + b^*$.

Let us split $2c^*$ into $c^* + c^*$. Then from (24) we can express c^*

$$\begin{aligned} 2c^* = & 2\mathcal{B}_{Y_M}(M^c, M^a) + (a+b)\mathcal{B}_{Y_F}(M^c, M^a) \\ & + 2\mathcal{B}_{Y_M}(M^c, M^b) + (a+b)\mathcal{B}_{Y_F}(M^c, M^b). \end{aligned}$$

Expanding the right hand side and recalling the definition of a^* and b^* in (22),(23) yields

$$2c^* = 2a^* + 2b^* + (b-a)\mathcal{B}_{Y_F}(M^c, M^a - M^b).$$

From (22),(24), we have

$$\begin{aligned} c^* = & 2\mathcal{B}_{Y_M}(M^c, M^a) + (a+b)\mathcal{B}_{Y_F}(M^c, M^a) \\ = & 2\mathcal{B}_{Y_M}(M^c, M^a) + 2a\mathcal{B}_{Y_F}(M^c, M^a) + (b-a)\mathcal{B}_{Y_F}(M^c, M^a) \\ = & 2a^* + (b-a)\mathcal{B}_{Y_F}(M^c, M^a). \end{aligned}$$

In a similar manner, we have the last equation from (23),(24).

Lemma 5 shows that it is enough to approximate the bilinear form $\mathcal{B}_{Y_F}(\cdot,\cdot)$ in order to estimate c^*. From the weak form for M^a and M^c in (17), we have

$$\begin{cases} \mathcal{B}_{Y_M}(M^c, M^c - M^a) + \dfrac{a+b}{2}\mathcal{B}_{Y_F}(M^c, M^c - M^a) = 0, \\ \mathcal{B}_{Y_M}(M^a, M^c - M^a) + a\mathcal{B}_{Y_F}(M^a, M^c - M^a) = 0. \end{cases} \tag{25}$$

Subtracting the above equations yields

$$\mathcal{B}_{Y_M}(M^c - M^a, M^c - M^a) = a\mathcal{B}_{Y_F}(M^a, M^c - M^a) - \frac{a+b}{2}\mathcal{B}_{Y_F}(M^c, M^c - M^a).$$

Since the left hand side is positive, expanding the right hand side implies

$$a\mathcal{B}_{Y_F}(M^a, M^a) + \frac{a+b}{2}\mathcal{B}_{Y_F}(M^c, M^c) < \frac{3a+b}{2}\mathcal{B}_{Y_F}(M^c, M^a).$$

Similarly, we get the following inequality

$$b\mathcal{B}_{Y_F}(M^b, M^b) + \frac{a+b}{2}\mathcal{B}_{Y_F}(M^c, M^c) < \frac{a+3b}{2}\mathcal{B}_{Y_F}(M^c, M^b).$$

Then these inequalities imply

$$\mathcal{B}_{Y_F}(M^a, M^c) > \frac{2a}{3a+b}\mathcal{B}_{Y_F}(M^a, M^a), \tag{26}$$

$$\mathcal{B}_{Y_F}(M^b, M^c) > \frac{2b}{a+3b}\mathcal{B}_{Y_F}(M^b, M^b). \tag{27}$$

Now, in the following theorem, we show the three different approximations of c^* by using a^*, b^*, and the auxiliary terms.

Theorem 6 *Suppose $b > a$. Then we have*

$$
\begin{aligned}
(i)\ & 0 \le c^* - (2a^* + (b-a)\delta_a) \le (b-a)\left(\sqrt{\frac{2\alpha b^*}{a+b}} - \delta_a\right),\\
(ii)\ & 0 \le (2b^* - (b-a)\delta_b) - c^* \le (b-a)\left(\sqrt{\frac{2\beta b^*}{a+b}} - \delta_b\right),\\
(iii)\ & P(\delta_a - T) \le c^* - (a^* + b^* + P(\delta_a - \delta_b)) \le P(T - \delta_a),
\end{aligned}
$$

where

$$
\begin{aligned}
&\alpha = ||\nabla M^a||^2_{L^2(Y_F)}, \quad \beta = ||\nabla M^b||^2_{L^2(Y_F)},\\
&\delta_a = \frac{2a}{3a+b}\alpha, \quad \delta_b = \frac{2b}{a+3b}\beta, \quad P = \frac{b-a}{2}, \quad T = \frac{2(b^*-a^*)}{b-a}.
\end{aligned}
$$

Since $\mathcal{B}_{Y_F}(M^c, M^a) > 0$, $\mathcal{B}_{Y_F}(M^c, M^b)$, and $b > a$, it is clear that $2a^* < c^* < 2b^*$ from (22) – (24). From (24), we have

$$
\frac{c^*}{2} = \mathcal{B}_{Y_F}(M^c, M^\eta) + \frac{a+b}{2}\mathcal{B}_{Y_F}(M^c, M^\eta), \quad \eta = a, b, c \tag{28}
$$

For $\eta = c$, we have

$$
||\nabla M^c||^2_{L^2(Y_F)} \le \frac{c^*}{a+b}. \tag{29}
$$

From the lemma 5 we express c^* in terms of a^* and $\mathcal{B}_{Y_M}(M^c, M^a)$. By applying the Hölder's inequality, together with (29) and $c^* < 2b^*$ we obtain the following estimate

$$
\begin{aligned}
c^* - 2a^* &= (b-a)\int_{Y_F} \nabla M^c \nabla M^a\\
&\le (b-a)||\nabla M^c||_{L^2(Y_F)}||\nabla M^a||_{L^2(Y_F)}\\
&\le (b-a)\sqrt{\frac{c^*\alpha}{a+b}}\\
&\le (b-a)\sqrt{\frac{2\alpha b^*}{a+b}}.
\end{aligned}
$$

By subtracting $(b-a)\delta_a$ from the both sides of the above inequality, we obtain the first inequality of the Lemma from (26). the second inequality is also obtained from (27) in a similar manner.

From (26) and (27), we have the following inequality for $B_{Y_F}(M^c, M^a - M^b)$

$$\delta_a - \mathcal{B}_{Y_F}(M^c, M^b) < \mathcal{B}_{Y_F}(M^c, M^a - M^b) < \mathcal{B}_{Y_F}(M^c, M^a) - \delta_b.$$

Then, from the (i) in Lemma 5,

$$P(\delta_b - \mathcal{B}_{Y_F}(M^c, M^b)) < c^* - (a^* + b^* + P(\delta_a - \delta_b)) < P(\mathcal{B}_{Y_F}(M^c, M^a) - \delta_a) \quad (30)$$

Let us recall the representation for a^*, b^* and c^* in $(22) - (24)$, then

$$\mathcal{B}_{Y_F}(M^c, M^b) = \frac{2b^* - c^*}{b - a}, \quad \mathcal{B}_{Y_F}(M^c, M^a) = \frac{c^* - 2a^*}{b - a} \quad (31)$$

Hence, the last inequality is proven by $(30), (31)$ with $2a^* < c^* < 2b^*$.

Remark 7 *The last approximation in Theorem* 6 *is actually the same as the average of the another two approximations.*

3 Effective modulus for the linear elasticity

Similarly, let us assume that a reference cell Y consists of two isotropic material such that $Y = Y_M \cup Y_F, \Gamma = \partial Y_M \cap \partial Y_F$ and $|Y| = 1$. Let $D(y) = [D_{ijkh}(y)]_{1 \le i,j,k,h \le N}$ be the elastic modulus tensor that is an $Y-$periodic. Let us note that for the isotropic material, the elastic modulus tensor is depend on a young's modulus, $E(y)$ and a poisson's ratio, $\nu(y)$. Assume that the poisson's ration, $\nu(y)$ is independent of y which is $\nu(y) \equiv \nu$ and the young's modulus, $E(y)$ varies. That is, $E(y) = E^M$ if $y \in Y_M$ and $E(y) = E^F$ if $y \in Y_F$. Then the elastic modulus tensor $D(y) = E(y)[a_{ijkh}(\nu)]$. From the theory of Homogenization in [4],[3], the effective modulus tensor D^* can be obtain by

$$D^*_{ijkh} = \frac{1}{|Y|} \int_Y \left(D_{ijpq}(y) \frac{\partial \chi_p^{kh}}{\partial y_q} + D_{ijkh} \right) dY, \quad (32)$$

where χ_p^{kh} is a $Y-$periodic solution of the following problem :

$$-\frac{\partial}{\partial y_j} \left(D_{ijpq} \frac{\partial \chi_p^{kh}}{\partial y_q} \right) = \frac{\partial D_{ijkh}}{\partial x_j} \quad \text{in Y}, \quad i = 1, \ldots, N, \quad \text{and} \quad \langle \chi_p^{kh} \rangle = 0. \quad (33)$$

Assume that χ_p^{kh} also satisfies the continuous interface condition along the Γ.

Now, for any $k, h \in \{1, \ldots, N\}$, let us define the vector valued function $P^{kh}(y) = (P_l^{kh})_{1 \le l \le N}$ by

$$P_l^{kh} = y_h \delta_{kl}, \quad l = 1, \ldots, N.$$

and let $\omega^{kh} = \chi^{kh} + P^{kh}$. Then, the variational form of (33) is as follows: Find $\chi^{kh} \in (W_{per}^1(Y))^N$ such that

$$\int_Y [e(\omega^{kh})]^T D(y)[e(v)]dy = 0, \quad \text{for all } v \in (W_{per}^1(Y))^N, \tag{34}$$

where e is the strain tensor defined by

$$e(\phi) = (e_{ij})_{1 \le i,j \le N}, \quad e_{ij}(\phi) = \frac{1}{2}\left(\frac{\partial \phi_i}{\partial y_j} + \frac{\partial \phi_j}{\partial y_i}\right), \quad \phi = (\phi_1, \ldots, \phi_N).$$

Moreover, the effective modulus D^* can be obtained by

$$D_{ijkh}^* = \frac{1}{|Y|} \int_Y [e(\omega^{ij})]^T D(y)[e(\omega^{kh})]dy, \tag{35}$$

Let us consider a normalized young modulus on Y_M. Then, from (35) together with the continuous interface condition, the effective modulus tensor D^* can be written by

$$D_{ijkh}^* = \mathcal{A}_{Y_M}(\omega^{ij}, \omega^{kh}) + E^F \mathcal{A}_{Y_F}(\omega^{ij}, \omega^{kh}), \tag{36}$$

where

$$\mathcal{A}_Y(u, v) = \int_Y [e(u)]^T A(y)[e(v)]dy, \quad A = [a_{ijkh}]. \tag{37}$$

Let us consider the diagonal components for D_{ijkh}^*. That is ,D_{ijij}^* only. For the notation, let us denote D_{ijij}^* and ω_d^{ij} by the effective modulus tensor and the vector valued function associated with a young modulus E_d in Y_F, respectively. Then D_{ijij}^* can be written by

$$D_{ijij}^* = \mathcal{A}_{Y_M}(\omega_d^{ij}, \omega_d^{ij}) + E_d \mathcal{A}_{Y_F}(\omega_d^{ij}, \omega_d^{ij}). \tag{38}$$

Since $\omega_d^{ij} - \omega_f^{ij} \in (W_{per}^1(Y))^N$, from (34) and (36), we have

$$D_{ijij}^* = \mathcal{A}_{Y_M}(\omega_d^{ij}, \omega_f^{ij}) + E_d \mathcal{A}_{Y_F}(\omega_d^{ij}, \omega_f^{ij}). \tag{39}$$

Remark 8 *If the diagonal components for D_{ijkh}^* are considered, then D_{ijij}^* has the same form for the effective coefficient in* (16). *This allows us to extend most properties for the effective coefficient to the effective modulus.*

Lemma 9 *For given ν, if $E_g \ge E_f$, then $G_{ijij}^* > F_{ijij}^*$, for all $i, j = 1, \ldots, N$.*

Let F^*_{ijkh} and G^*_{ijkh} be the effective moduli associated with the material property, $E_f[a_{ijkh}(\nu)]$ and $E_g[a_{ijkh}(\nu)]$ on Y_F, respectively. Let D^*_{ijkh} be the effective modulus associated with the material property $E_d[a_{ijkh}]$ on Y_F, where $E_d = E_f + E_g$. In the same argument in the previous section, the diagonal components of G^*_{ijkh} can be approximated in terms of the diagonal components of D^*_{ijkh} and F^*_{ijkh}. For simplicity, let us denote F^*_{ijij}, G^*_{ijij} and D^*_{ijij} by F^*, G^* and D^*, respectively. Let us ω^{ij}_d denoted by ω_d.

Theorem 10 *Suppose $E_g > E_f$ and ν is fixed. Then we have*

$$(i)\ 0 \le D^* - (2F^* + (E_g - E_f)\delta_a) \le (E_g - E_f)\left(\sqrt{\frac{2\alpha F^*}{E_f + E_g}} - \delta_a\right),$$

$$(ii)\ 0 \le (2G^* - (E_g - E_f)\delta_b) - D^* \le (E_g - E_f)\left(\sqrt{\frac{2\beta G^*}{E_f + E_g}} - \delta_b\right),$$

$$(iii)\ P(\delta_a - T) \le D^* - (F^* + G^* + P(\delta_a - \delta_b)) \le P(T - \delta_a),$$

where

$$\alpha = \mathcal{A}_{Y_F}(\omega_d, \omega_d), \quad \beta = \mathcal{A}_{Y_F}(\omega_f, \omega_f),$$
$$\delta_a = \frac{2E_f}{3E_f + E_g}\alpha, \quad \delta_b = \frac{2E_g}{E_f + 3E_g}\beta, \quad P = \frac{E_g - E_f}{2}, \quad T = \frac{2(G^* - F^*)}{E_g - E_f}.$$

4 Numerical Results

In this section, we present some numerical results to support the theoretical predictions All numerical results presented in this section are for two dimensional problems. However, the results can be extended to the three dimensional problems without loss of generosity.

4.1 Periodic boundary condition

In order to find the effective coefficient or modulus, the cell problems in (7) or (33) must be solved with the periodic boundary condition. However, it is not clear how to assign the periodic boundary condition in a numerical computation. In this section, appropriate periodic boundary conditions are demonstrated.

For the cell problem in (7), since χ_p is the $Y-$periodic funtion, the periodic

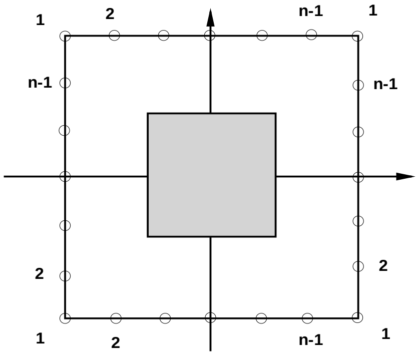


Fig. 2. Periodic boundary condition.

condition can be expressed by

$$\chi_p(y) = \chi_p(y + Y), \quad \forall y \in Y. \tag{40}$$

This leads to the periodic boundary condition;

$$\chi_p(y_0) = \chi_p(y_0 + Y), \quad \forall y_0 \in \partial Y. \tag{41}$$

There are several ways to assign the periodic boundary condition: For the symmetric fiber inclusion, the periodic boundary condition leads to the specific boundary condition. Bakhvalov and Panasenko used the mixed dirchlet and neumann boundary conditions on the boundary in [1]. That is, $\chi_1(\pm 1/2, y_2) = \pm 1/2$ and $\dfrac{\partial \chi_1}{\partial y_2}(y_1, \pm 1/2) = 0$. For non-symmetric fiber inclusion, the appropriate boundary conditions are imposed. Conca and Natesan used the conjugate gradient method to solve the discrete system by assigning an average of the values on the boundary at each iteration in [2].

We assigned the periodic boundary condition such that for each nodal point corresponding to an apposite edge, it will be considered to be the same nodal point. Figure 2 shows the periodic boundary condition, In other words,

$$\begin{aligned} \chi_p(y_i) &= \chi_p(y_{i+m}), \quad y_i \in \partial Y, \quad i = 1, \cdots, m, \\ \chi_p(y_j) &= \chi_p(y_{j+n}), \quad y_i \in \partial Y, \quad i = 1, \cdots, n. \end{aligned} \tag{42}$$

If the m by n-many nodal points are used to solve the cell problem, the actual degree of freedom is $(m-1) \times (n-1)$. Since this constraint seems to be the neumann boundary condition, $\chi_p(0,0)$ is set to be a zero for the unique solution.

For the cell problem of the linear elasticity in (33), there are similar approaches to assign the periodic boundary condition as stated before. It is

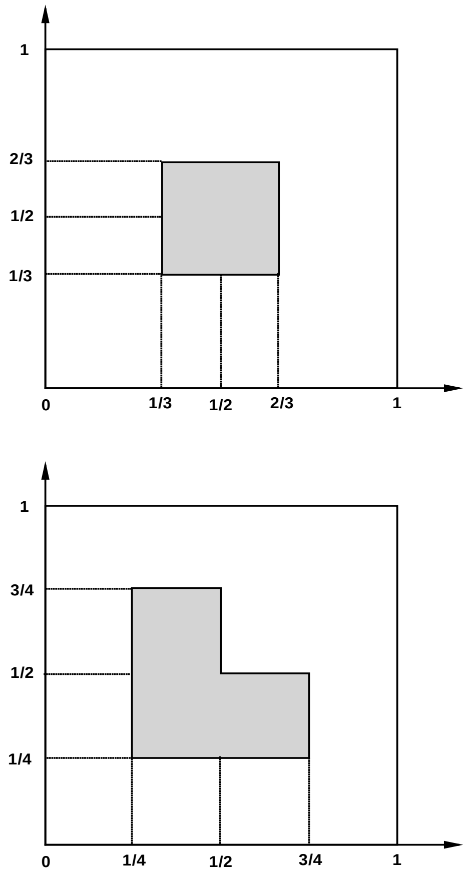


Fig. 3. Square symmetric(left) and non-symmetric(right) fiber inclusion reference cells.

referred to see [6],[8] and [9] for the details.

4.2 Effective coefficients in the various reference cells

For all numerical results, we consider the normalized material property in Y_M. Let $[\kappa_{ij}(y)]$ be the material property matrix such that

$$\kappa_{ij}(y) = \begin{cases} \delta_{ij}, & y \in Y_M, \\ \kappa\delta_{ij}, & y \in Y_F, \end{cases}$$

Example 5.1 First, we compare the effective coefficients by assigning different periodic boundary conditions on symmetric and non-symmetric fiber inclusion in Y as given Fig. 3.

The compared effective coefficients with the average boundary values [2] and the direct periodic nodal constraint in (42) are given in Tables 1 and 2 for

Table 1
Comparisons of Effective coefficients in symmetric fiber inclusion.

Degrees of Freedom(DOF)	κ	Average of the values	Periodic constraint
1800	10	1.2158	1.2120
	1/18	0.8196	0.8093

Table 2
Comparisons of Effective coefficients in non-symmetric fiber inclusion.

κ	DOF	Average of the values	DOF	Periodic constraint
10	1152	1.4278	1064	1.4228
1/114		0.6429		0.6383

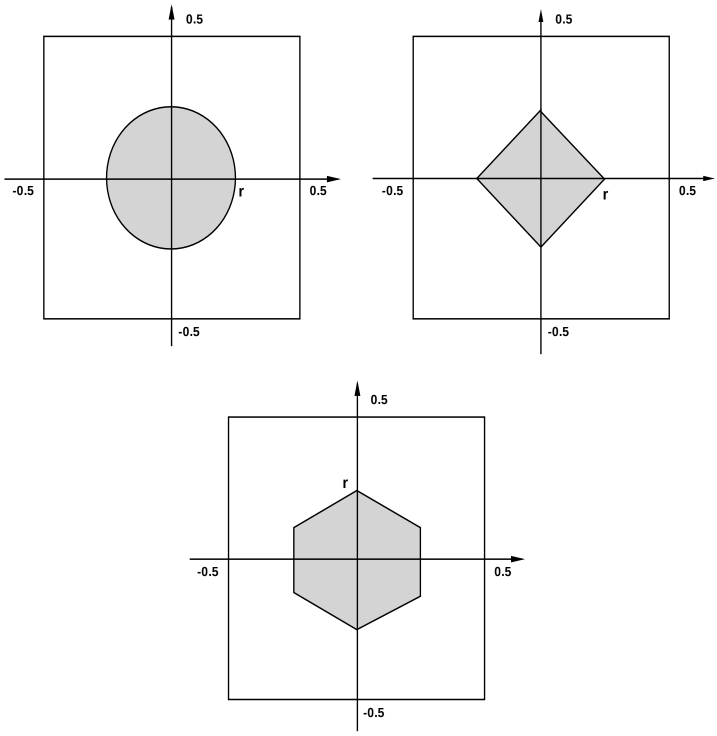


Fig. 4. A different shape fiber inclusion: circular, rhombus and hexagon.

the various values of κ. As seen in [2], the effective coefficients are decreased as the degrees of freedom are increased for the given κ. From this observation, the effective coefficient with the direct periodic nodal constraint is more accurate in the same degrees of freedom.

Remark 11 *For the symmetric fiber inclusion in* $Y = (-1/2.1/2)$, *the effective coefficient is independent of any periodic boundary conditions.*

In the following examples, the effective coefficients for the various values of κ are shown. In the case of the symmetric fiber inclusion, the mixed dirch-

Table 3
Effective coefficients $\kappa^* = \kappa^*_{11} = \kappa^*_{22}$ for various values of κ in a non-symmetric fiber inclusion in a square reference cell.

κ	Effective coefficient κ^*	κ	Effective Coefficient κ^*
0.001	0.6312	2	1.1360
0.01	0.6391	10	1.4218
0.05	0.6712	20	1.4985
0.1	0.7059	100	1.5785
0.5	0.8804	1000	1.5998

let and neumann boundary conditions are assigned as the periodic boundary condition. In the cases of the non-symmetric fiber inclusions, the direct nodal constraint is applied as the periodic boundary condition.

Example 5.2 Let us consider the non-symmetric fiber inclusion with the same dimension as shown in Figure 1. The effective coefficients in this reference cell are shown in Table 3. The values of the effective coefficients are plotted in Figure 5 when the values of κ vary from 0.001 to 1000. The numerical results shows that the effective coefficient is increasing as the value of κ is increasing.

Example 5.3 In this example, several symmetric fiber inclusions are considered with the constant area being 1/4: circular, hexagon and rhombus. The effective coefficients for each shape of the finer inclusion are shown in Tables 5, 6 and 4, respectively. Comparisons of the effective coefficients with the different fiber inclusion are plotted in Figure 6. It is also shown that the effective coefficient is increasing as the material property κ is increasing. Moreover, it is observed that the shape of the fiber inclusion influenced the value of effective coefficient depending on the value of κ even if the area is fixed. For $0.5 \leq \kappa \leq 5$, the magnitude of the effective coefficient are almost independent of the shape of the fiber inclusion. However, for $\kappa \geq 5$, the magnitude of the effective coefficient is decreasing as the shape of the fiber inclusion is close to the circle. For $\kappa \leq 0.5$, the value of the effective coefficient is increasing as the shape of the fiber inclusion is close to the circle.

4.3 Effective elastic moduli in the various reference cells

All numerical examples presented in this section are under plane strain condition. Poisson's ratio are set to be 0.3 in a reference cell, Y=$(-1/2, 1/2)^2$. Let E^M and E^F be the young's modulus on Y_M and Y_F, respectively. Assume that $E^M = 1$. The explicit boundary conditions [6] are applied as the periodic boundary condition.

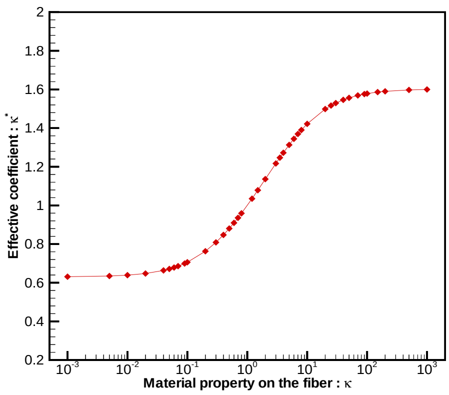


Fig. 5. Effective coefficients for the values of κ on the non-symmetric fiber inclusion.

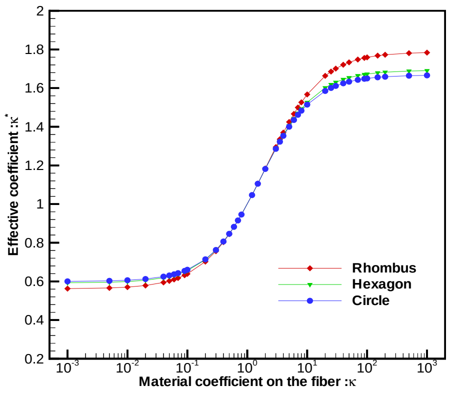


Fig. 6. Comparison of Effective coefficients for the values of κ on the symmetric fiber inclusion with the constant area being 1/4.

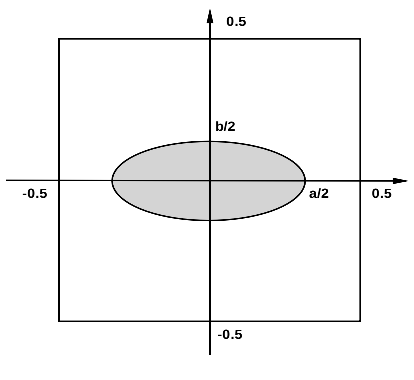


Fig. 7. Unit reference cell with a ellipse fiber inclusion.

Table 4
Effective coefficients κ^* for various values of κ in a symmetric rhombus fiber inclusion with $a = 1/(2\sqrt{2})$.

κ	Effective coefficient κ^*	κ	Effective Coefficient κ^*
0.001	0.5626	2	1.1842
0.01	0.5702	10	1.5672
0.05	0.6024	20	1.6634
0.1	0.6388	100	1.7591
0.5	0.8444	1000	1.7836

Table 5
Effective coefficients κ^* for various values of κ in a symmetric hexagon fiber inclusion with $a^2 = 1/(6\sqrt{3})$.

κ	Effective coefficient κ^*	κ	Effective Coefficient κ^*
0.001	0.5929	2	1.1823
0.01	0.5990	10	1.5260
0.05	0.6251	20	1.6018
0.1	0.6558	100	1.6727
0.5	0.8457	1000	1.6902

Table 6
Effective coefficients κ^* for various values of κ in a symmetric circular fiber inclusion with radius $a = 1/(2\sqrt{\pi})$.

κ	Effective coefficient κ^*	κ	Effective Coefficient κ^*
0.001	0.6002	2	1.1818
0.01	0.6059	10	1.5148
0.05	0.6307	20	1.5853
0.1	0.6601	100	1.6501
0.5	0.8461	1000	1.6659

Example 5.4 In order to demonstrate the accuracy of effective modulus, we compare our results with those results [7] by the Boundary Element Method(BEM). Let Y_F be a elliptic fiber inclusion with $a = 0.3785$ and $b = 0.2523$, which implies that the aspect ratio a/b is 1.5. Table 7 shows the comparisons between the effective moduli of the P-version of FEM and the single-region BEM. All results agree up to 2-digits.

Example 5.5 In this example, we investigate the behavior of effective elastic moduli in elliptic fiber inclusions for the choice of the ratio of aspect. The

Table 7
Comparisons between the effective moduli of the P-version of FEM(PFEM) and the single-region Boundary Element Method(S-BEM)

E^F	D^*_{1111}		D^*_{1122}		D^*_{1212}		D^*_{2222}	
	PFEM	S-BEM	PFEM	S-BEM	PFEM	S-BEM	PFEM	S-BEM
0.01	.7009	0.7000	.1644	.1625	0.4713	0.4720	.1155	.1172
0.1	.7944	0.7942	.2424	.2418	0.6288	0.6297	.1670	.1679
10	.2793	2.2792	.7910	.7901	1.9644	1.9659	.5524	.5528
100	.5876	2.5854	.8088	.8082	2.0793	2.0807	.5816	.5830

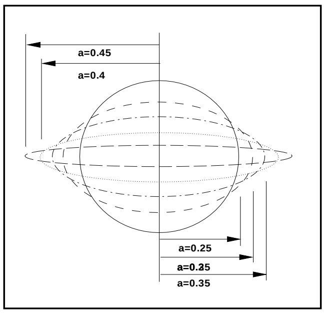
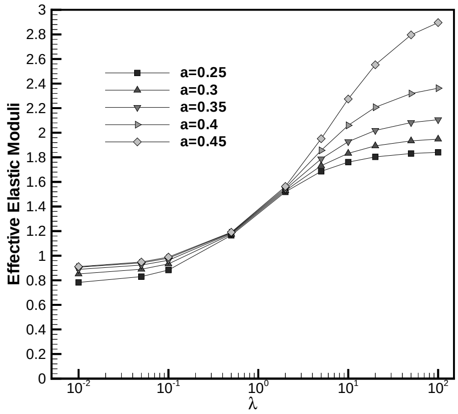
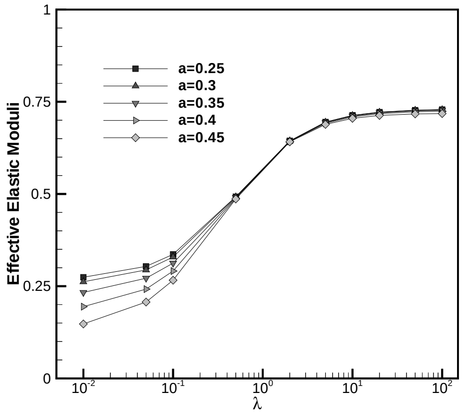


Fig. 8. An ellipse fiber inclusion for different ratios of aspect.

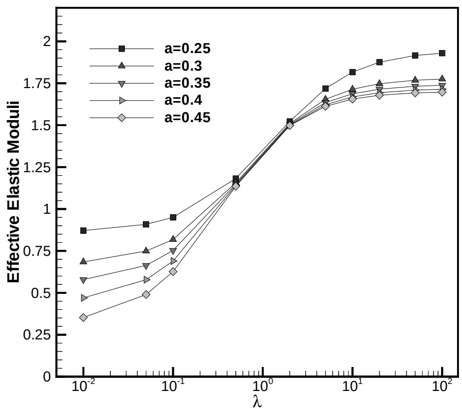
area of the elliptic fiber Y_F is set to be the same as the area of the circle with radius $r = 0.25$. Here, we choose four different major diameter a, which are $0.3, 0.35, 0.4, 0.45$. Figure 9 shows the behavior of the effective elastic moduli. From Figure 9, we have the following observations of the effective elastic moduli:

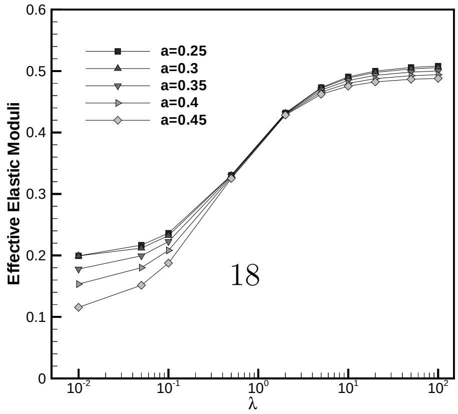
(1) All values of effective elastic moduli increase as the value of E^F increases, which supports Lemma 9.
(2) All effective elastic moduli are almost no variations for the choice of the major diameter a when $0 < E^F < 5$.
(3) For D^*_{1122} and D^*_{1212}, they have similar behaviors through the choice of the value of a. In particular, they decrease as the major diameter a increases for $E^F < 1$.
(4) D^*_{1111} and D^*_{2222} behave in the opposite way. That is, if $E^F > 1$, then D^*_{1111} increases as the major diameter a increases. If $E^F > 1$, then D^*_{1111} decreases as the major diameter a increases. However, in case of D^*_{2222}, the opposite behavior was observed.





(a) (b)





(c) (d)

Fig. 9: Behaviour of the Effective Elastic Moduli for four choices of the

4.4 The Approximation of Effective Coefficients for the sum of two material properties

In this section, we give the numerical examples of the approximated effective coefficients for the sum of two material properties. Let us assume that the reference cell is the circular fiber inclusion with a radius being 0.25 in $\mathrm{Y}=(-1/2, 1/2)^2$. In order to demonstrate the accuracy of three approximating formulas stated in Theorem 6, we introduce the following notations:

- $E(a^M, a^F)$ = the effective coefficient with respect to the a^M of the matrix parts Y_M and a^F of the fiber parts Y_F, respectively.
- $A_k(a^M + b^M, a^F + b^F)$ = an approximation of $E(a^M + b^M, a^F + b^F)$. Here k stands for each of three formulas listed in Theorem 6.

For examples, for the normalized material constants,

$$E(1, 3) + E(1, 7) \approx A_k(2, 10) \approx E(2, 10) = 2 \cdot E(1, 5).$$

Moreover, the effective coefficient $E(2, 10)$ can be approximated by many different combinations as follows:

$$E(2, 10) \approx A_k(1 + 1, 2 + 8), A_k(1 + 1, 4 + 6), etc.$$

In the Table 8 we present the approximations of effective coefficients for two possible combinations, $A_k(2, a_1^F + b_1^F)$ and $A_k(2, a_2^F + b_2^F)$ by using three methods in Theorem 6, where $a_1^F/b_1^F \gg a_2^F/b_2^F \approx 1$. From the numerical results in the Table 8, we have the following observations in $A_k(2, a^F + b^F)$:

(1) It follows from Lemma 1 that for $a^F = b^F$, $E(1, a^F)+E(1, b^F) = E(2, a^F + b^F)$. When $a^F/b^F \approx 1$, all approximated values $A_k(2, a^F + b^F)$ is very accurate.

(2) Since $A_1(2, a^F + b^F) \leq E(2, a^F + b^F)$ and $A_2(2, a^F + b^F) \geq E(2, a^F + b^F)$, the average value of $A_1(\cdot, \cdot)$ and $A_2(\cdot, \cdot)$ may be the better approximation. In fact, $A_3(\cdot, \cdot)$ is the best approximation compared with other approximations for almost every choice of a^F and b^F.

(3) From all results, it is concluded that $A_3(\cdot, \cdot) \leq E(\cdot, \cdot)$. This implies that $A_2(\cdot, \cdot)$ is the better approximation than $A_1(\cdot, \cdot)$ because $A_3(\cdot, \cdot)$ is the average value of $A_1(\cdot, \cdot)$ and $A_2(\cdot, \cdot)$.

Table 8
Comparisons of exact $E(2, a^F + b^F)$ with approximated effective coefficients $A_k(2, a^F + b^F)$, $\lambda = a^F + b^F$ in a circular reference cell with r=0.25.

λ	$E(2,\lambda)$	(a^F, b^F)	A_1	A_2	A_3
100	2.9305	$(10, 90)$	2.8888	2.9437	2.9163
		$(40, 60)$	2.9254	2.9347	2.9300
30	2.8301	$(5, 25)$	2.7471	2.8641	2.8056
		(10, 20)	2.8027	2.8494	2.8260
10	2.6026	$(2, 8)$	2.4514	2.6712	2.5613
		$(4, 6)$	2.5701	2.6290	2.5995
5	2.3675	$(1.5, 3.5)$	2.2656	2.4327	2.3491
		$(2, 3)$	2.3245	2.4025	2.3635
1	1.7542	$(0.2, 0.8)$	1.6100	1.8288	1.7214
		$(0.4, 0.6)$	1.7198	1.7824	1.7510
0.6	1.6174	$(0.1, 0.5)$	1.4944	1.6773	1.5858
		$(0.2, 0.4)$	1.5710	1.6505	1.6170
0.1	1.3964	$(0.01, 0.09)$	1.3613	1.4113	1.3863
		$(0.04, 0.06)$	1.3906	1.4010	1.3958

4.5 The Approximation of Effective Elastic Moduli for the sum of two material properties

In this section, we give the numerical examples of the approximated effective moduli for the sum of two material properties. All results are under plane strain condition with the Poisson's ratio being set to be 0.3 in the same reference cell stated in the previous section.

In order to demonstrate the accuracy of three approximating formulas stated in Theorem 10, we introduce the similar notations in the previous section:

- $D_{ijij}(E^M, E^F)$ = the effective elastic muduli with respect to the Young's modulus E^M of the matrix parts Y_M and E^F of the fiber parts Y_F, respectively.
- $A^k_{ijij}(E^M_1 + E^M_2, E^F_1 + E^F_2)$ = an approximation of $D_{ijij}(E^M_1 + E^M_2, E^F_1 + E^F_2)$. Here k stands for each of three formulas listed in Theorem 10.

For $a^*_{1111} = a^*_{2222}$

λ	$E(2,\lambda)$	(E_A^F, E_B^F)	A_3	(a^F, b^F)	A_3
100	3.6624	$(10, 90)$	3.6519	$(40, 60)$	3.6619
40	3.6074	$(5, 25)$	3.5587	$(10, 30)$	3.5898
10	3.3724	$(0.1, 0.5)$	1.6599	$(0.2, 0.4)$	1.6456
5	3.1342	$(1.5, 3.5)$	3.1144	$(2, 3)$	3.1298
1	2.3312	$(0.2, 0.8)$	2.2706	$(0.4, 0.6)$	2.3260
0.6	2.0988	$(0.1, 0.5)$	2.0377	$(0.2, 0.4)$	2.0856
0.1	1.6598	$(0.01, 0.09)$	1.6373	$(0.04, 0.06)$	1.6587

λ	$E(2,\lambda)$	(E_A^F, E_B^F)	A_3	(a^F, b^F)	A_3
100	1.0118	$(10, 90)$	1.0097	$(40, 60)$	1.0117
40	0.9998	$(5, 25)$	0.9904	$(10, 30)$	0.9963
10	0.9464	$(0.1, 0.5)$	1.6599	$(0.2, 0.4)$	1.6456
5	0.8882	$(1.5, 3.5)$	0.8826	$(2, 3)$	0.8870
1	0.6612	$(0.2, 0.8)$	0.6415	$(0.4, 0.6)$	0.6598
0.6	0.5824	$(0.1, 0.5)$	0.5828	$(0.2, 0.4)$	0.5822
0.1	0.4337	$(0.01, 0.09)$	0.4253	$(0.04, 0.06)$	0.43328

For a^*_{1212}

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