Basic ideas of differential transform method

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Many problems in science and engineering fields can be described by the partial differential equations. A variety of numerical and analytical methods have been developed to obtain accurate approximate and analytic solutions for the problems in the literature.

The classical Taylor series method is one of the earliest analytic technique to many problems, specially ordinary differential equations. However, since it requires a lot of symbolic calculation for the derivatives of functions, it takes a lot of computational time for higher order derivatives. Here, we introduce the updated version of the Taylor series method which is called the differential transform method(DTM). The DTM is the method to determine the coefficients of the Taylor series of the function by solving the induced recursive equation from the given differential equation. The basic idea of the DTM was introduced by Zhou [14].

In what following we introduce a few notations for the DTM. Suppose that the solution $u(\mathbf{x},t)$ is analytic at $(\widetilde{\mathbf{x}},\widetilde{t})$, then the solution $u(\mathbf{x},t)$ can be represented by the Taylor series

$$u(\mathbf{x},t)=\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k_1!\cdots k_n!h!}\left[\frac{\partial^{k_1+\cdots+k_n+h}u(\widetilde{\mathbf{x}},\widetilde{t})}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}\partial t^h}\right]\left(\prod_{i=1}^{n}(x_i-\widetilde{x_i})^{k_i}\right)(t-\widetilde{t})^h. \quad (1)$$

Definition 1 Let us define the $(n+1)$ dimensional differential transform $U(\mathbf{k},h)$ of $u(\mathbf{x},t)$ at $(\widetilde{\mathbf{x}},\widetilde{t})$ by

$$U(\mathbf{k},h)=\frac{1}{k_1!\cdots k_n!h!}\left[\frac{\partial^{k_1+\cdots+k_n+h}u(\widetilde{\mathbf{x}},\widetilde{t})}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}\partial t^h}\right].$$

Definition 2 The differential inverse transform of $U(\mathbf{k},h)$ is defined by $u(\mathbf{x},t)$ of the form in (1).

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Thus, $u(\mathbf{x},t)$ can be written by

$$u(\mathbf{x},t)=\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}\sum_{h=0}^{\infty}U(\mathbf{k},h)\left(\prod_{i=1}^{n}(x_i-\widetilde{x_i})^{k_i}\right)(t-\widetilde{t})^h.$$

It is noted that the differential transform $U(\mathbf{k},h)$ is the nothing but the coefficient of the Taylor series of $u(\mathbf{x},t)$. However, the DTM is different to the classics Taylor series method in determining the coefficients of the Taylor series. In order to obtain the differential transform, the DTM provides a recursive equation which is derived by collecting the coefficients of the Taylor series with the order of $\left(\prod_{i=1}^{n}(x_i-\widetilde{x_i})^{k_i}\right)(t-\widetilde{t})^h$ for all functions in the given differential equation. Thus, it is a key to obtain a recursive equation corresponding to the given differential equation. In many work [2–5,9,14], appropriate recursive equations were presented and well performances have been achieved in many important problems in science and engineering fields. Some fundamental operations for the three dimension DTM are presented in Table 1. For example, let us consider the following initial problem:

$$\frac{\partial u(x,y,t)}{\partial t}=u(x,y,t)+f(x,y,t). \tag{2}$$

Using the basic properties of the DTM in Table 1 implies the following recursive equation:

$$(h+1)U(k_1,k_2,h+1)=U(k_1,k_2,h)+F(k_1,k_2,h), \tag{3}$$

where $F(k_1,k_2,h)$ is the differential transform of $f(x,y,t)$. Suppose $f(x,y,t)=\Delta u=\partial^2u/\partial x^2+\partial^2u/\partial y^2$, then the corresponding differential transform is

$$F(k_1,k_2,h)=(k_1+1)(k_1+2)U(k_1+2,k_2,h)+(k_2+1)(k_2+2)U(k_1,k_2+2,h).$$

From the differential transform $U(k_1,k_2,0)$ of the given initial condition $u(\mathbf{x},0)$, the recursive equation (3) can be easily solved. Let us recall other analytic methods such as the Adomian decomposition [1,5,10–13] and the variation iteration[6–8] that determine a successive component by integrating a previous component. It is easy to see that those methods encounter difficulties in calculating integration for the complicate function. But, the DTM does not require calculation of such integration. It is only required to solve a algebraic recursive equation. Thus, it is very effective method to solve the many linear and nonlinear partial differential equations.

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Table 1
Fundamental operations for the three dimensional DTM

Original function $w(x,y,t)$	Transformed function $W(k,h,m)$
$u(x,y,t) \pm v(x,y,t)$	$U(k,h,m) \pm V(k,h,m)$
$cu(x,y,t)$	$cU(k,h,m)$
$\dfrac{\partial^{r+s+p} u(x,y,t)}{\partial x^r \partial y^s \partial t^p}$	$\dfrac{(k+r)!}{k!}\dfrac{(h+s)!}{s!}\dfrac{(m+p)!}{p!}U(k+r,h+s,m+p)$
$\dfrac{\partial u(x,y,t)}{\partial x}\dfrac{\partial v(x,y,t)}{\partial y}$	$\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(k-r+1)(h-s+1)U(k-r+1,s,p)U(r,h-s+1,m-p)$
$u(x,y,t)v(x,y,t)$	$\sum_{r=0}^{j}\sum_{s=0}^{h}\sum_{p=0}^{m}U(r,h-s,m-p)V(k-r,s,p)$
$u(x,y,t)v(x,y,t)g(x,y,t)$	$\sum_{r=0}^{k}\sum_{\tau=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{l=0}^{m}\sum_{q=0}^{m-l}U(r,h-s-p,l)V(\tau,s,m-l-q)G(k-r-\tau,p,q)$

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